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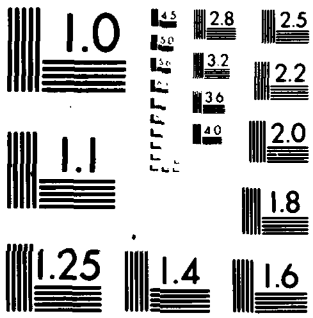

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WEAPONS SYSTEMS RESEARCH LABORATORY

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TECHNICAL MEMORANDUM

WSRL-0139-TM

ON THE DISTRIBUTION OF SIGNAL-TO-NOISE RATIO WHEN ESTIMATED FROM A POWER SPECTRUM

A.P. CLARKE

Technical Memoranda are of a tentative nature, representing the views of the author(s), and do not necessarily carry the authority of this Laboratory.

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9 TECHNICAL MEMO~~RANDUM~~

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SUMMARY

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POSTAL ADDRESS: Chief Superintendent, Weapons Systems Research Laboratory, Box 2151, G.P.O., Adelaide, South Australia, 5001.

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A rigorous derivation is given of a closed form analytic expression for the probability density function of signal-to-noise power ratio when estimated from a power spectrum containing a noisy signal tone. Formulae are given for the mean and variance and sufficient information is also supplied to enable higher order moments to be derived. The derivation of the above formulae are followed by brief considerations of the implications in experimental measurement design.

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TABLE OF CONTENTS

LIST OF APPENDICES

LIST OF FIGURES

1. INTRODUCTION

As part of a study of acoustic processing techniques it was necessary to make estimates from experimental data of the ratio of received signal power to noise power per hertz. One method is to record at a sensor a sinusoidal tone emitted by a distant source. The received signal is then digitised and the power spectrum computed from a set of data samples. The sample rate and the number of data samples are chosen so that an appropriate frequency bin of the spectrum will contain the total signal power apart from side lobes of negligible height. This signal bin will also contain a component of noise power. Two alternatives are now possible:

First, an estimate (z) of the ratio of signal to noise-per-hertz can be obtained from the single spectrum by evaluating

$$z = \frac{s^2 - u}{ru} \tag{1}$$

or

$$z = \frac{s^2}{ru} - \frac{1}{r} \tag{2}$$

where s^2 is the power including noise measured in the signal bin.

u is the average power in frequency bins near the signal bin but not containing extraneous signals.

$\frac{1}{r}$ is the bin width in hertz.

Secondly, a number of spectra from consecutive sets of data samples can be obtained. An average spectrum can then be formed and an estimate of z can be obtained from using either equation (1) or (2).

In this paper a probabilistic analysis is presented for the first alternative. A number of assumptions were found to be necessary to arrive at a satisfactory result;

(1) The received signal tone has constant amplitude during the time over which an estimate of signal-to-noise ratio is made.

(2) The number of spectral bins used for spectral analysis is large. e.g. the number of samples used is $\geqslant 1024$.

(3) The noise field as recorded by the sensor is isotropic, stationary, continuous and white.

2. THE DISTRIBUTION OF z WHEN ESTIMATED FROM A SINGLE SPECTRUM

The power spectrum of a sampled time series is the output from a digital algorithm and consists of a tabular sequence of power estimates for each frequency bin.

Following Rice(ref.1) the probability density function of the envelope of a signal tone in narrow band noise is given by

$$f_S(x) = 2A^2 x \exp(-A^2 (x^2+K^2)) \; I_0(2A^2 Kx), \; x \geqslant 0 \tag{3}$$

where K is the amplitude of the signal tone

$\frac{1}{2A^2}$ is the mean square noise amplitude

I_0 is the modified Bessel function of zero order.

The noise can be assumed to have zero mean and hence the noise variance is $\frac{1}{2A^2}$.

The density function for the envelope of narrow band noise is obtained by putting K equal to zero in equation (3), i.e.

$$f_N(x) = 2A^2 x \exp(-A^2 x). \tag{4}$$

The estimator for signal-to-noise ratio is

$$z = \frac{1}{r}\left(\frac{s^2}{\frac{1}{p}\Sigma\, n_i^2}\right) - \frac{1}{r} \tag{5}$$

where s^2 is the power in the frequency bin containing the signal tone

n_i^2 is the power in any bin that contains noise only

p is the number of bins used to estimate the average noise power

$\frac{1}{r}$ is the bin width in hertz.

Therefore, the estimator (z) is the ratio of the signal tone power to noise-power-in-a-1 Hz-band.

As the bins have a width that is generally small in comparison with the signal bin centre frequency, equations (3) and (4) can be used to derive the distributions of s^2 and n_i^2 in equation (5).

Generally, for probability density functions where $f_X(x) = 0$ for $x < 0$

$$f_{X^2}(x) = \frac{1}{2x^{\frac{1}{2}}} f_X(x^{\frac{1}{2}}). \tag{6}$$

This can be found in Parzen(ref.3).

Hence

$$f_{S^2}(x) = A^2 \exp(-A^2(x + K^2))\ I_0(2A^2Kx^{\frac{1}{2}}) \tag{7}$$

and

$$f_{N_i^2}(x) = A^2 \exp(-A^2 x). \tag{8}$$

Equation (8) is the density of a function distributed as $\chi^2_{2,1/A2^{\frac{1}{2}}}$.

In Appendix I it is shown that u, where

$$u = \frac{1}{p}\sum_{i=1}^{p} n_i^2, \tag{9}$$

is distributed as $\chi^2_{2p,1/(A2^{\frac{1}{2}}p^{\frac{1}{2}})}$; i.e.

$$f_U(x) = \frac{1}{2^p\left(\frac{1}{A\sqrt{2p}}\right)^{2p}\Gamma(p)} x^{p-1} \exp\left\{\frac{-x}{2\left(\frac{1}{A\sqrt{2p}}\right)^2}\right\}. \quad (10)$$

The probability density function for the ratio s^2/u can now be obtained by applying the classical relationship given in Parzen for the density of the ratio of 2 independent positive random variables

$$f_{X/Y}(y) = \int_o^\infty x\ f_X(yx)\ f_Y(x)\ dx \quad (11)$$

i.e.

$$f_{S^2/U}(y) = \int_o^\infty dx\ x\ A^2 \exp(-A^2(xy + K^2))\ I_0(2A^2K\sqrt{xy})$$

$$\times \frac{A^{2p}p^p}{\Gamma(p)} x^{p-1} \exp(-A^2px) \quad (12)$$

$$= \frac{A^{2p+2}p^p \exp(-A^2K^2)}{\Gamma(p)} \int_o^\infty dx\ x^p \exp(-A^2x(y+p)) \quad I_0(2A^2K\sqrt{xy}). \quad (13)$$

For ease in writing put

$$\alpha = \frac{A^{2p+2}p^p \exp(-A^2K^2)}{\Gamma(p)} \quad (14)$$

$$\beta = A^2(y + p) \quad (15)$$

$$\gamma = 2A^2K\sqrt{y}. \quad (16)$$

Then

$$f_{S^2/U}(y) = \alpha \int_o^\infty dx\ x^p \exp(-\beta x)\ I_0(\gamma\sqrt{x}) \quad (17)$$

Making the substitution $u = \sqrt{x}$, $2u\ du = dx$
then

$$f_{S^2/U}(y) = 2\alpha \int_o^\infty du\ u^{2p+1} \exp(-\beta u^2)\ I_0(\gamma u). \quad (18)$$

Using (9.6.3) of Abramowitz and Stegun(ref.2), this equation can be rewritten as

$$f_{S^2/U}(y) = 2\alpha \int_0^\infty du\ u^{2p+1} \exp(-\beta u^2)\ J_0(i\gamma u). \tag{19}$$

The integral can be simplified and evaluated by replacing the Bessel function with its series expansion and reversing the order of integration and summation. The resulting integral in the general term can be evaluated using the integration formula (7.4.5) in Abramowitz and Stegun. Hence, after some rearrangement

$$f_{S^2/U}(y) = \frac{\alpha}{\beta^{p+1}} \sum_{j=0}^{\infty} \frac{\Gamma(p+j+1)}{(j!)^2}\left(\frac{\gamma^2}{2^2\beta}\right)^j . \tag{20}$$

Rewriting the Gamma function in terms of Pochhammers symbol defined for example in (6.1.22) of Abramowitz and Stegun gives

$$f_{S^2/U}(y) = \frac{\alpha}{\beta^{p+1}}\,\Gamma(p+1) \sum_{j=0}^{\infty} \frac{(p+1)_j}{(j!)^2}\left(\frac{\gamma^2}{4\beta}\right)^j . \tag{21}$$

i.e.

$$f_{S^2/U}(y) = \frac{\alpha\,\Gamma(p+1)}{\beta^{(p+1)}}\, M\!\left(p+1,\ 1,\ \frac{\gamma^2}{4}\right). \tag{22}$$

where M(., ., .) is the Confluent Hypergeometric function known as Kummer's Function and whose series expansion is given for example in (13.1.2) of Abramowitz and Stegun. Substitution from equations (14), (15) and (16) now enables the density function to be written in terms of its primary parameters as

$$f_{S^2/U}(y) = \frac{p^{p+1}}{(y+p)^{p+1}} \exp(-A^2K^2)\ M\!\left(p+1,\ 1,\ \frac{A^2K^2y}{y+p}\right). \tag{23}$$

The distribution function for z now follows from the relation

$$f_{aX+b}(y) = \frac{1}{a} f_X\!\left(\frac{y-b}{a}\right)$$

i.e.

$$f_Z(z) = \frac{rp^{p+1}}{(rz+p+1)^{p+1}} \exp(-A^2K^2)\ M\!\left(p+1,\ 1,\ \frac{A^2K^2(rz+1)}{rz+p+1}\right) \tag{24}$$

where $-\frac{1}{r} < z < \infty$.

Typical shapes taken by this distribution are shown in figure 1. These were derived using the simple program given in Appendix II.

3. THE MEAN AND VARIANCE OF z

The mean and variance of z are derived readily from the mean and second moment of S^2/U. The moments of S^2/U are

$$E(y^n) = \int_0^\infty dy\, y^n \frac{p^{p+1}}{(y+p)^{p+1}} \exp(-A^2K^2)\, M\left(p+1,\ 1,\ \frac{A^2K^2y}{y+p}\right) \tag{25}$$

Substitution of the series expansion for M(., ., .) and altering the order of integration and summation gives rise to an integral in the general summation term which, on evaluation, forms a β-function.

The β-function can then be written in terms of Γ-functions. After some algebraic manipulation:

$$E(y^n) = p^{n+1} \exp(-A^2K^2)\, \beta(p-n,n+1)\, M(n+1,\ 1,\ A^2K^2). \tag{26}$$

The details of the derivation are given in Appendix III.
The mean of y is now

$$E(y) = p^2 \exp(-A^2K^2)\, \beta(p-1,2)\, M(2,\ 1,\ A^2K^2). \tag{27}$$

From (13.4.1) and (13.6.12) in Abramowitz and Stegun

$$M(2,\ 1,\ A^2K^2) = (A^2K^2+1) \exp(A^2K^2).$$

From (6.2.2) in Abramowitz and Stegun

$$\beta(p-1,2) = \frac{\Gamma(p-1)\ \Gamma(2)}{\Gamma(p+1)} = \frac{1}{p(p-1)}.$$

Hence after some rearranging

$$E(y) = \frac{(A^2K^2+1)p}{(p-1)}. \tag{28}$$

The second moment, derived in a similar fashion, is

$$E(y^2) = p^3 \exp(-A^2K^2)\, \beta(p-2,3)\, M(3,1,A^2K^2). \tag{29}$$

From (13.4.1) and (13.6.12) in Abramowitz and Stegun

$$M(3,1,A^2K^2) = \tfrac{1}{2}(2 + 4A^2K^2 + A^4K^4) \exp(A^2K^2).$$

From (6.2.2) in Abramowitz and Stegun

$$P(p-2,3) = \frac{\Gamma(p-2)\ \Gamma(3)}{\Gamma(p+1)} = \frac{2}{p(p-1)(p-2)}.$$

After some rearranging

$$E(y^2) = \frac{p^2(2 + 4A^2K^2 + A^4K^4)}{(p-1)\ (p-2)} . \qquad (30)$$

The mean and second moment of z now follow from application of the properties of the expection operator E to the equation

$$z = \frac{1}{r} y - \frac{1}{r}$$

i.e.

$$E(z) = \frac{1}{r} E(y) - \frac{1}{r}$$

$$= \frac{p(1 + A^2K^2)}{r(p-1)} - \frac{1}{r} \qquad \text{from equation (28)} \qquad (31)$$

Again

$$E(z^2) = \frac{1}{r^2} E(y^2) - \frac{2}{r^2} E(y) + \frac{1}{r^2} .$$

The variance of z is then

$$\sigma^2(z) = E(z^2) - (E(z))^2$$

$$= \frac{1}{r^2} E(y^2) - \frac{2}{r^2} E(y) + \frac{1}{r^2} - \frac{1}{r^2} (E(y))^2$$

$$+ \frac{2}{r^2} E(y) - \frac{1}{r^2}$$

i.e.

$$\sigma^2(z) = \frac{1}{r^2} E(y^2) - \frac{1}{r^2} (E(y))^2 .$$

Substitution from equations (28) and (30) gives after some rearrangement

$$\sigma^2(z) = \frac{p^2(p + 2A^2K^2p + A^4K^4)}{r^2(p - 1)^2(p - 2)} . \qquad (32)$$

4. THE LIMITING DISTRIBUTION OF z IN THE ABSENCE OF A SIGNAL TONE

When the signal has zero amplitude, the signal-to-noise ratio is zero and the probability density function and associated mean and variance are simply obtained by putting A^2K^2 to zero in equations (24), (31) and (32) respectively.
Hence the probability density function has the form

$$f_Z(z) = \frac{p^{p+1}}{(rz + p + 1)^{p+1}} \tag{33}$$

where $\frac{-1}{r} \leqslant z < \infty$.

The mean is given by

$$E(z) = \frac{1}{r(p - 1)}, \text{ and} \tag{34}$$

the variance is given by

$$\sigma^2(z) = \frac{p^3}{r^2(p - 1)^2(p - 2)}. \tag{35}$$

5. THE LIMITING DISTRIBUTION OF z WHEN THE INPUT SIGNAL-TO-NOISE RATIO IS LARGE

For large signal-to-noise ratio the relation (13.1.4) in Abamowitz and Stegun can be used to rewrite $f_Z(z)$ as expressed by equation (24) to give

$$f_Z(z) = \frac{p^{p+1}}{(rz + p + 1)^{p+1}} \exp(-A^2K^2) \frac{1}{(p+1)} \frac{(rz+1)^p}{(rz+p+1)^p} \exp\left\{\frac{-A^2K^2(rz+1)}{(rz+p+1)}\right\}$$

i.e.

$$f_Z(z) = \frac{p^{p+1}(rz+1)^p}{p!(rz+p+1)^{2p+1}} \exp\left\{\frac{-A^2K^2p}{rz+p+1}\right\}. \tag{36}$$

The mean and variance follow simply from equations (31) and (32), i.e.

$$E(z) = \frac{pA^2K^2}{r(p-1)} \tag{37}$$

and

$$\sigma^2(z) = \frac{p^2A^4K^4}{r^2(p-1)^2(p-2)}. \tag{38}$$

6. DESIGN OF EXPERIMENTS

In order to use the preceding analysis in the design of experiments, the coefficient of variation of the distribution (equation (24)) should be examined for selected values of r and p.

The coefficient of variation is defined as

$$C = \frac{\sigma}{E(z)}$$

and is a measure of the dispersion of a distribution function when the expected value is the unit of measurement.

From equations (31) and (32)

$$C = \frac{p}{pA^2 K^2 + 1} \left(\frac{p + 2pA^2 K^2 + A^4 K^4}{p - 2} \right)^{\frac{1}{2}} .$$

A typical behaviour of C is shown in figure 2. It would appear that any improvement in handling low signal-to-noise ratio by using higher values of p is not possible.

An alternative way of looking at the design of experiments is shown in figure 3. Here the bias in the mean of the measurement is clearly seen. Also, the problem in the long right-hand tail of the distribution giving an increased chance of obtaining negative values for the estimate of signal-to-noise ratio is evident. This latter problem is also illustrated by equation (34) and (35) where $E(z) - \sigma$ is always negative and equations (37) and (35) where $E(z) - \sigma$ is always positive.

7. CONCLUSION

An analysis has been presented of a frequently used estimator for signal-to-noise ratio. The result in closed form is new as far as the author can determine. A simple technique for examining any proposed experiment is then described and points to the inability to measure signal-to-noise ratio with any precision at low valuès of the input signal-to-noise ratio. This points to the necessity of examining the use of multiple estimates of the power spectrum to obtain an average spectrum from which an estimate of signal-to-noise ratio can be made.

Some progress has already been made with this problem and it is proposed to publish a paper on this topic shortly.

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No.	Author	Title
1	Rice, S.O.	"Mathematical Analysis of Random Noise!". Bell System Tech. J., July 1944
2	Abramowitz, M. and Stegun, I.	Handbook of Mathematical Functions, Dover Publications, 1970
3	Parzen, E.	"Modern Probability Theory and its Application". John Wiley & Sons, New York, 1960

APPENDIX I

THE DISTRIBUTION OF $u = \frac{1}{p}\sum_{1}^{p} n_i^2$

If x is distributed as $\chi^2_{n,\sigma}$ then x has a probability density function

$$f_X(x) = \frac{1}{2^{n/2}\,\sigma^n\,\Gamma(n/2)}\, x^{n/2-1} \exp\left(\frac{-x}{2\sigma^2}\right).$$

Hence $\chi^2_{2,\sigma}$ has a probability density function

$$f_X(x) = \frac{1}{2\sigma^2} \exp\left\{\frac{-x}{2\sigma^2}\right\}.$$

The addition of p samples from $\chi^2_{2,\sigma}$ is equivalent to adding 2p squared terms where each term is a sample from $N(0, \sigma)$; i.e. the sum of p samples from $\chi^2_{2,\sigma}$ is distributed as $\chi^2_{2p,\sigma}$ and the probability density function of the sum is given by

$$f_X(x) = \frac{1}{2^p\,\sigma^{2p}\,\Gamma(p)}\, x^{p-1} \exp\left\{\frac{-x}{2\sigma^2}\right\}.$$

The probability density function of the average is found using the general relation (Parzen)

$$f_{aX}(x) = \frac{1}{a} f_X\left(\frac{x}{a}\right)$$

i.e.

$$f_U(x) = \frac{p}{2^p\,\sigma^{2p}\,\Gamma(p)}\, p^{p-1}\, x^{p-1} \exp\left(\frac{-px}{2\sigma^2}\right)$$

$$= \frac{1}{2^p\left(\frac{\sigma}{\sqrt{p}}\right)^{2p}\Gamma(p)}\, x^{p-1} \exp\left\{\frac{-x}{2\left(\frac{\sigma}{\sqrt{p}}\right)^2}\right\}$$

i.e. U is distributed as $\chi^2_{2p,\sigma/\sqrt{p}}$.

APPENDIX II

EVALUATION OF PROBABILITY DENSITY FUNCTION EQUATION (4)

NOTE THAT THE CONFLUENT HYPERGEOMETRIC FUNCTION IS EVALUATED RECURSIVELY IN PROCEDURE FUNC USING EQUATION (13.4.1) IN HANDBOOK OF MATHEMATICAL FUNCTIONS BY ABRAMOWITZ AND STEGUN, DOVER PRESS, 1970

```
SNRPDF:  PROCEDURE OPTIONS (MAIN):

         Z = -1;

         R =  1;    /*   1/R = FREQUENCY BIN WIDTH

         IP = 18;   /*   IP  = NO OF BINS OVER WHICH NOISE IS AVERAGED

         Y = 2;     /*   Y   = INPUT SNR IN DECIBELS

         K = 10**(Y/10);

         DO I = 1 TO 50;

         A = K*(R*Z+1)/(R*Z+IP+1);

         B = FUNC(IP,1,A);

         C = ((IP/(R*Z+IP+1))**(IP+1))*EXP(-K)*B;

         PUT EDIT (Z,C)(SKIP(1),F(2),X(3),F(12,6));

         Z = Z + 1;   END;   END;

FUNC, PROCEDURE (IP,1,X);

         M1 = EXP(X);   M2 = (2+X)*EXP(X);

         DO I = 3 TO IP;

         M3 = ((2*(I-1)-1+X)/(I-1)*M2 - (1-(1/(I-1)))*M1;

         M1 = M2;   M2 = M3;   END;

         RETURN(M3);         END FUNC;

         END;
```

APPENDIX III

DERIVATION OF EQUATION (26)

$$E(y^n) = \int_0^\infty dy\ y^n \frac{p^{p+1}}{(y+p)^{p+1}} \exp(-A^2K^2)\ M\left(p+1,\ 1,\ \frac{A^2K^2y}{y+p}\right)$$

$$= p^{p+1} \exp(-A^2K^2) \int_0^\infty \frac{y^n}{(y+p)^{p+1}} \sum_0^\infty \frac{\Gamma(p+1+j)}{\Gamma(p+1)(j!)^2} \left\{\frac{A^2K^2y}{(y+p)}\right\}^j dy$$

$$= p^{p+1} \exp(-A^2K^2) \sum_0^\infty \frac{\Gamma(p+1+j)\,(A^2K^2)^j}{\Gamma(p+1)\ \ (j!)^2} \int_0^\infty \frac{y^{n+j}}{(y+p)^{p+j+1}}\, dy$$

$$= p^{n+1} \exp(-A^2K^2) \sum_0^\infty \frac{\Gamma(p+1+j)\,(A^2K^2)^j}{\Gamma(p+1)\ \ (j!)^2}\ \beta(p-n,n+j+1)$$

$$= p^{n+1} \exp(-A^2K^2) \sum_0^\infty \frac{\Gamma(p+1+j)\,(A^2K^2)^j}{\Gamma(p+1)\ \ (j!)^2}\ \frac{\Gamma(p-n)\ \Gamma(n+j+1)}{\Gamma(p+1+j)}$$

$$= p^{n+1} \exp(-A^2K^2) \frac{\Gamma(p-n)\ \Gamma(n+1)}{\Gamma(p+1)} \sum_0^\infty \frac{\Gamma(n+j+1)\,(A^2K^2)^j}{\Gamma(n+1)\ \ (j!)^2}$$

$$= p^{n+1} \exp(-A^2K^2)\ \beta(p-n,n+1)\ M(n+1,1,A^2K^2)$$

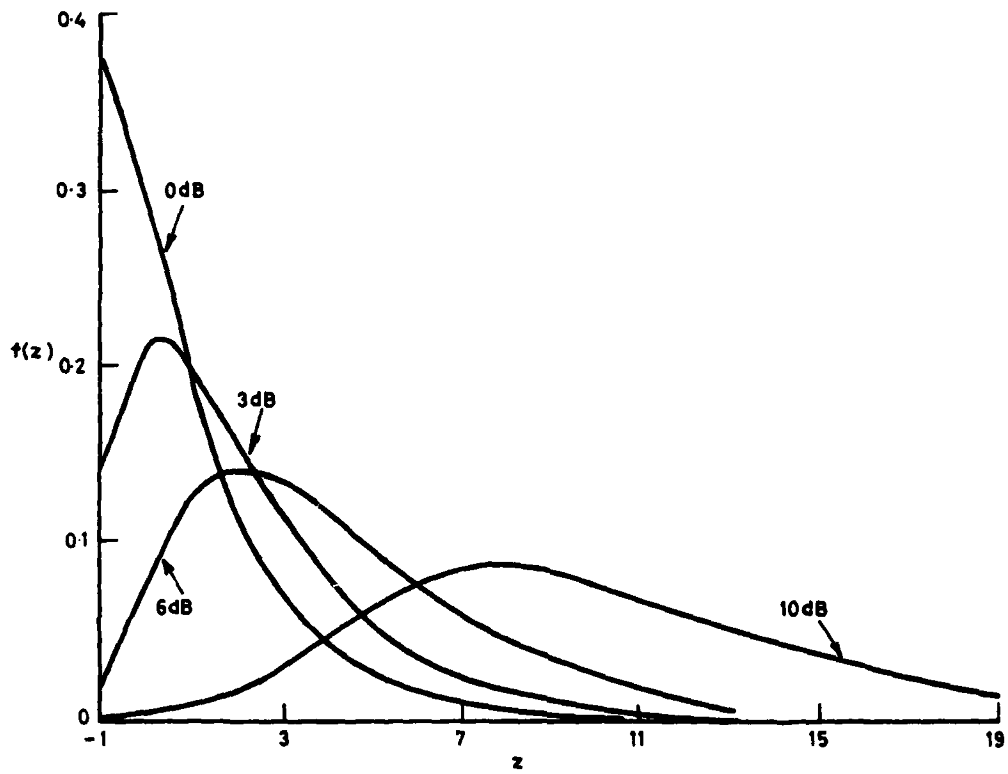

Figure 1. Probability density functions for r = 1, p = 18 and input signal-to-noise ratios of 0 dB, 3 dB, 6 dB and 10 dB

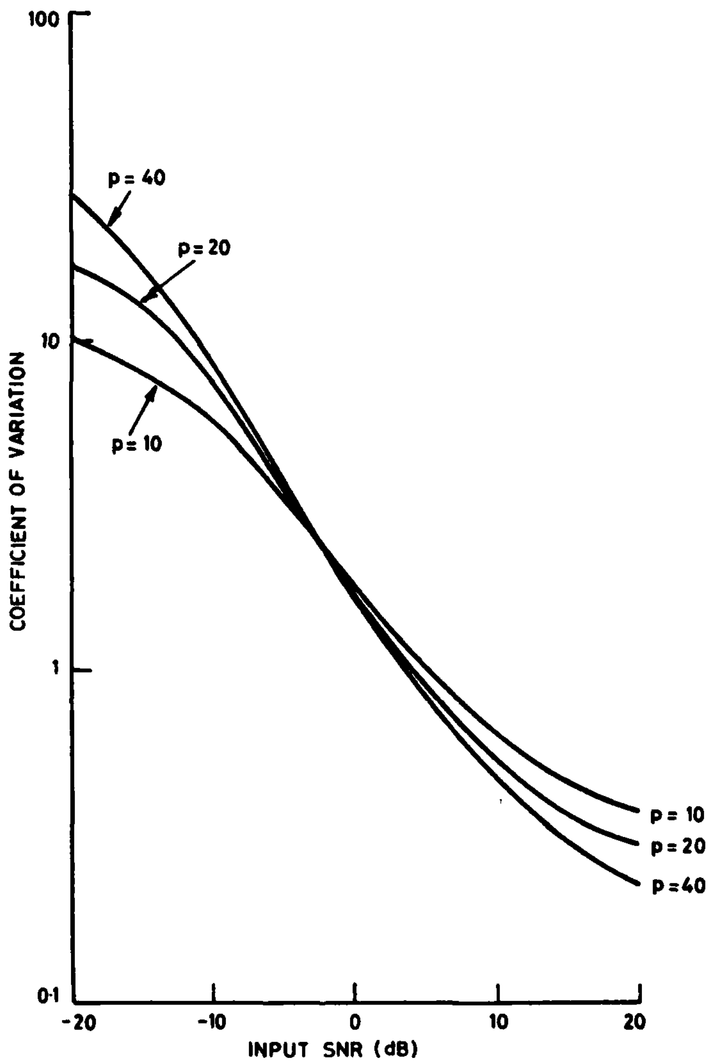

Figure 2. Coefficient of variation as a function of input signal-to-noise ratio for $r = 1$ and $p = 10$, 20 and 40

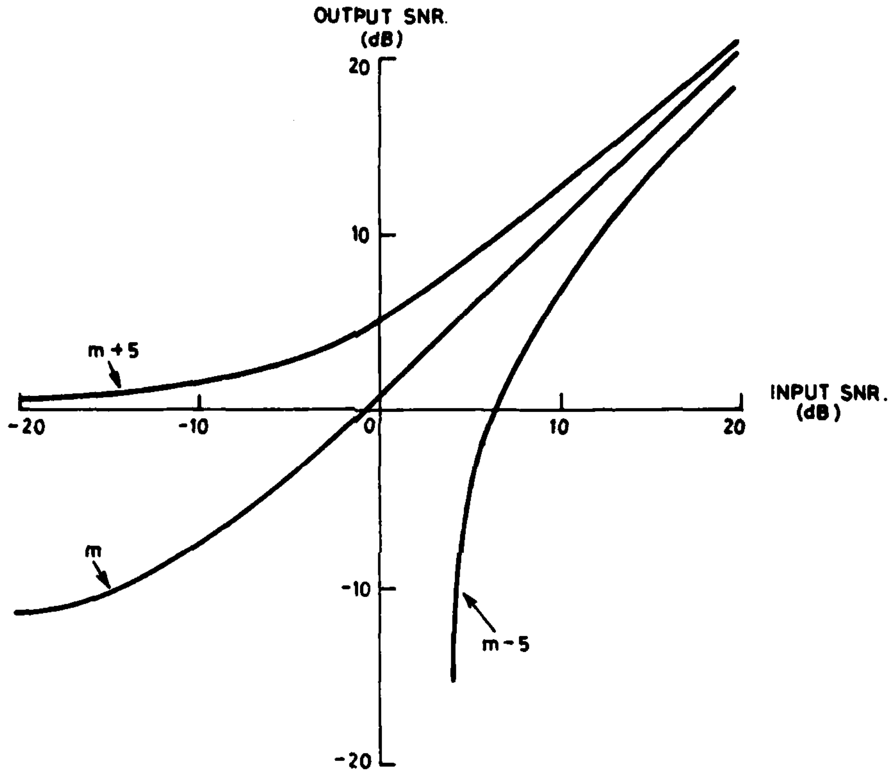

Figure 3. Variation in the output signal-to-noise ratio as a function of the input signal-to-noise ratio for p = 18, r = 1

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Dr A. Cantoni, Department Electrical Engineering, University of Newcastle	10
Chief Defence Scientist	11
Deputy Chief Defence Scientist	12
Superintendent, Science and Technology Programmes	13
Director, Joint Intelligence Organisation (DDSTI)	14
Superintendent, RAN Research Laboratory	15
Defence Library, Campbell Park	16
Library, Aeronautical Research Laboratories	17
Library, Materials Research Laboratories	18
Defence Information Services Branch (for microfilming)	19
Defence Information Services Branch for:	
United Kingdom, Ministry of Defence, Defence Research Information Centre (DRIC)	20
United States, Department of Defense, Defense Documentation Center	21 - 32
Canada, Ministry of National Defence, Defence Science Information Service	33
New Zealand, Ministry of Defence	34
Australian National Library	35

	Copy No.
WITHIN DRCS	
Chief Superintendent, Weapons Systems Research Laboratory	36
Chief Superintendent, Electronics Research Laboratory	37
Superintendent, Electronic Warfare Division	38
Superintendent, Weapon Systems Division	39
Superintendent, Radar Division	40
Principal Officer, Cybernetic Electronics Group	41
Principal Officer, Microwave Radar Group	42
Senior Principal Research Scientist, Marine Studies	43
Dr M. Lees, Jindalee Development Group	44
Principal Officer, Underwater Detection Group	45
Principal Officer, Marine Physics Group	46
Dr D.A. Gray, Underwater Detection Group	47
Dr D. Kewley, Underwater Detection Group	48
Dr A.K. Steele, Underwater Detection Group	49
Dr D.G. Cartwright, Marine Physics Group	50
Author	51 - 52
DRCS Library	53 - 54
Spares	55 - 60

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-8